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LETTER TO THE EDITOR

Bose–Einstein condensation in geometrically deformed tubes

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Abstract
In this letter we discuss a new physical effect: we show that the Bose–Einstein condensate can be created in quasi-one-dimensional systems in a purely geometrical way, namely by bending or other suitable deformation of a tube. We demonstrate this for the perfect Bose gas as well as for the case of mean-field particle interactions.

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It is well known [1] that there is no Bose–Einstein condensate (BEC) of the continuous free Bose gas in one dimension as well, and that the same is true in tube-shaped domains $D \times \mathbb{R}$ with a compact cross section $D \subset \mathbb{R}^2$. On the other hand, there is increasing experimental and theoretical interest [2–4] in the BEC in quasi-one-dimensional systems of trapped boson gases, in particular with the idea of understanding the properties predicted by the Lieb–Liniger model [5–7] and the Girardeau–Tonks gas [8–10].

In this letter we want to draw attention to a way to create BEC in a quasi-one-dimensional perfect Bose gas (PBG) based on *geometrical properties* of the recipient[4]. Dealing with a PBG we naturally have to describe first the one-particle spectrum. Consider first a *straight* cylindrical recipient, i.e. an infinitely long tube $\mathcal{C}(S_r)$ of a circular cross section of radius $r > 0$. The one-particle Hamiltonian is (up to a multiplicative constant) the Dirichlet Laplacian *t*. It can be defined through the associated quadratic form [12] or simply, since the tube boundary is smooth, as $T\psi = -\Delta\psi$ with the condition $\psi(x) = 0$ at the tube boundary. By separation of variables, the spectrum of T equals $\bigcup_{j=1}^{\infty}[E_j, \infty)$, where

$$0 < E_1 < E_2 \leqslant E_3 \leqslant \cdots \qquad (1)$$

[4] The recipient geometry will remain restricted here even in the thermodynamic limit in contrast to the recent study of the so-called anisotropic Casimir boxes [11].

are the eigenvalues of the two-dimensional operator $-(\partial_x^2 + \partial_y^2)$ with the Dirichlet boundary conditions at the edge of S_r. The lowest one is, of course, non-degenerate; for the circular cross section we can write explicitly $E_1 = \frac{\hbar^2}{2M} j_{0,1}^2 r^{-2}$, where $j_{0,1}$ is the first zero of $J_0(z)$, and the higher eigenvalues are similarly expressed through other Bessel function zeros. The integral density of states corresponding to the spectrum of t is given by

$$\mathcal{N}(\varepsilon) = [2\sqrt{\pi}\Gamma(3/2)]^{-1} \sum_{j=1}^{\infty} \theta(\varepsilon - E_j)\sqrt{\varepsilon - E_j}, \tag{2}$$

where θ is the Heaviside function. This gives for the PBG in the infinite tube the grand canonical total-particle density

$$\rho(\beta, \mu) = \int_0^{\infty} \mathcal{N}(\mathrm{d}\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} \tag{3}$$

provided we have temperature $\beta^{-1} \geqslant 0$ and chemical potential $\mu < E_1$. Since the critical value $\rho_c(\beta) := \lim_{\mu \nearrow E_1} \rho(\beta, \mu)$ of the density is infinite, there is no BEC of the PBG in this quasi-one-dimensional system.

Several possibilities are known to make the critical density finite, provoking thus a BEC, by changing the one-particle spectrum. One is to replace the Dirichlet boundary condition by a 'sticky' one, i.e. by a mixed condition with a positive outside gradient of the one-particle wavefunction on the boundary [13] (in fact, it is sufficient to make the replacement at the 'lids' of a finite cylinder). One can also switch in an external local attractive potential producing bound state(s) below the $\inf \sigma(T) = E_1$, the continuum spectrum [14, 15]. A less obvious way is a suppression of the density states at the bottom of the spectrum, leading to convergence of the integral (3) for $\mu = E_1$, by embedding the PBG into a random external potential [16].

In this letter we are going to show that a *geometrical deformation* of the straight tube such as a simple local bending, even a gentle one, may produce the BEC of the PBG with the condensate localized in the vicinity of such a bend. We will also discuss other types of (local) geometrical deformations creating the BEC in these quasi-one-dimensional systems as well the conditions under which the effect could be experimentally attainable.

Our claim is based on some recent results about one-particle spectra in deformed tubes which we will first briefly recall. Quantities referring to the deformed tubes will always be marked by an asterisk. Consider first a *local bend* of the infinite cylinder $\mathcal{C}(S_r)$, so that the tube axis is a smooth curve which is straight outside a compact region; we also suppose that the bent tube $\mathcal{C}^*(S_r)$ does not intersect itself. Whenever the deformation is nontrivial, $\mathcal{C}^*(S_r) \neq \mathcal{C}(S_r)$, it generates one or more eigenvalues [17, 18] below the continuum threshold E_1.

Their number is finite and the lowest one of them is simple; the form of this discrete spectrum is determined by the geometry of the deformed tube. Properties of such bound states in bent tubes are well understood:

- (a1) If the tube is only slightly bent there is only a bound state with energy ϵ_1^* and the gap $E_1 - \epsilon_1^*$ is proportional to φ^4 with a known coefficient, where φ is the bending angle of $\mathcal{C}^*$, see [18].
- (a2) The tube need not be straight outside a bounded region; it is sufficient that it is asymptotically straight in the sense that the tube axis curvature decays faster than $|s|^{-1}$ as $|s - s_0| \to \infty$ for a fixed s_0 where s is the arclength of the tube axis.
- (a3) The cross section need not be circular. In such a case, however, the shape is restricted by the so-called *Tang condition* imposed on the torsion [18]. It is satisfied, in particular, if $\mathcal{C}^* = \mathcal{S}^* \times [0, d]$, where $\mathcal{S}^*$ is a bent planar strip [19, 20].
- (a4) The effect is robust. It also does not require the tube $\mathcal{C}^*$ to be bent smoothly; the geometrically induced bound states exist also in sharply bent tubes [21, 22].

Furthermore, bending is not the only way to produce geometrically a nonempty discrete spectrum:

(b1) Another mechanism is a local change of the cross section. Protrusions and more general deformations which enlarge the volume also give rise to an effective attractive interaction [23].
(b2) A similar effect comes from a tube branching. A tube in the form of a right-angle cross is known to support a single bound state [24], and numerous isolated eigenvalues arise in a skewed scissor-shaped cross with small enough angle [25].
(b3) One more example is two parallel tubes with a window in the common boundary, where the bound state number is given by the window length [26].

Recall also that the geometrically induced discrete spectrum is *finite* in asymptotically straight ducts. In the bent-tube case, for instance, it follows from the fact that the effective attractive potential falls off faster than $|s|^{-2}$, see [18]. On the other hand, the actual form of the spectrum depends on the tube geometry. In particular, if we deal with two or more well-localized, identical and mutually distant perturbations the eigenvalues cluster around those of a single deformation, with a split exponentially small w.r.t. the distance between the perturbations. This is proved mathematically for a tube with a pair of windows [27]; however, various examples worked out numerically [28] suggest that such behaviour is generic.

Consider thus such a geometrically induced discrete spectrum below the continuum threshold in a deformed cylinder which consists of one or several bound states with energies $\{\epsilon_s^* < E_1 : s = 1, 2, \ldots\}$ and eigenfunctions localized in the vicinity of the deformation(s); recall that the threshold $\inf \sigma(T_*) = E_1$ is preserved as long as the tube remains asymptotically straight. If one orders the bound-state energies naturally, $\epsilon_1^* < \epsilon_2^* \leqslant \cdots < E_1$, then the domain of the allowed chemical potentials is given by the inequality $\mu \leqslant \epsilon_1^*$. This implies that the critical particle density is bounded,

$$\rho_c^*(\beta) := \lim_{\mu \nearrow \epsilon_1^*} \int_{E_1}^{\infty} \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} \, \mathrm{d}\mathcal{N}^*(\varepsilon) < \infty; \tag{4}$$

here $\mathcal{N}^*(\varepsilon)$ is the integrated density of states for the deformed cylinder. This opens a way to create the BEC.

To make this effect transparent consider first a finite segment of the deformed cylinder $\mathcal{C}_L^*$ of length L. Then the corresponding one-particle Hamiltonian is the Dirichlet Laplacian $T_*(L)$ in $L^2(\mathcal{C}_L^*)$ defined in the same way as above; since $\mathcal{C}_L^*$ is bounded it has a purely discrete spectrum $\sigma(T_*(L))$ accumulating at infinity. Denote by $P_I(T_*(L))$ the respective spectral measure of an interval $I \subset \mathbb{R}$. It allows us to write the finite-volume integrated density of states corresponding to the operator $T_*(L)$ as

$$\mathcal{N}_L^*(\epsilon) := \frac{1}{|\mathcal{C}_L^*|} \mathrm{Tr}\{P_{(-\infty,\epsilon)}(T_*(L))\}, \tag{5}$$

where $|\mathcal{C}_L^*|$ is, of course, the volume of the segment $\mathcal{C}_L^*$. By (5) the total particle density $\rho_L^*(\beta, \mu)$ in $\mathcal{C}_L^*$ acquires the form

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mathcal{N}_L^*(\varepsilon)}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} = \frac{1}{|\mathcal{C}_L^*|} \sum_{\{\epsilon_s^*(L)\}} \frac{1}{\mathrm{e}^{\beta(\epsilon_s^*(L)-\mu)} - 1} + \frac{1}{|\mathcal{C}_L^*|} \sum_{\{\varepsilon_j^*(L)\}} \frac{1}{\mathrm{e}^{\beta(\varepsilon_j^*(L)-\mu)} - 1}. \tag{6}$$

Here $\{\epsilon_s^*(L)\}_{s\geqslant 1}$ and $\{\varepsilon_j^*(L)\}_{j\geqslant 1}$ are eigenvalues of the operator $T_*(L)$ divided into two groups with $\epsilon_s^* < \varepsilon_j^*$ for any s, j. The first one consists of those which converge to the eigenvalues of the infinite-tube operator T_* as $L \to \infty$; recall that all of them are monotonically decreasing with respect to L. The limit is naturally taken in such a way that the distance of the cut-offs from the deformed part(s) tend to infinity. On the other hand, $\{\varepsilon_j^*(L)\}_{j\geqslant 1}$ are those eigenvalues which give in this limit the continuous spectrum of T_*. Combining this with the above-stated properties of the one-particle spectrum we see that the first group $\{\epsilon_s^*(L)\}_{s\geqslant 1}$ is finite. Since $|\mathcal{C}_L^*| \to \infty$ as $L \to \infty$, one gets then from (6) for the limiting density of states $\rho^*(\beta,\mu) := \lim_{L\to\infty} \rho_L^*(\beta,\mu)$ the relation

$$\rho^*(\beta,\mu) = \lim_{L\to\infty} \frac{1}{|\mathcal{C}_L^*|} \sum_{\{\varepsilon_j^*(L)\}} \frac{1}{\mathrm{e}^{\beta(\varepsilon_j^*(L)-\mu)}-1} = \int_{E_1}^{\infty} \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}-1}\,\mathrm{d}\mathcal{N}^*(\varepsilon), \qquad (7)$$

provided $\mu \leqslant \lim_{L\to\infty} \epsilon_1^*(L) = \epsilon_1^* < E_1$, uniformly in μ, where $\mathrm{d}\mathcal{N}^*(\varepsilon)$ is the (weak) limit of the 'cut-off' measure family $\{\mathrm{d}\mathcal{N}_L^*(\varepsilon)\}_L$. In particular, relation (7) implies

$$\rho^*(\beta,\mu) < \rho_c^*(\beta) := \rho^*(\beta,\epsilon_1^*) \qquad (8)$$

for $\mu < \epsilon_1^*$ in view of (4).

Now let the total particle density be $\rho > \rho_c^*(\beta)$. To show that this implies the BEC, let us analyse solutions $\{\mu_L(\beta,\rho)\}_L$ of the implicit equation following from (6),

$$\rho = \rho_L^*(\beta,\mu); \qquad (9)$$

it is easy to see that they always exist and are bounded by $\epsilon_1^*(L)$ from above. Using (6) and (7) one gets for $\rho \leqslant \rho_c^*(\beta)$ the limiting relation

$$\lim_{L\to\infty} \mu_L(\beta,\rho) = \mu(\beta,\rho) \leqslant \epsilon_1^*. \qquad (10)$$

On the other hand, for $\rho > \rho_c^*(\beta)$ we can rewrite equation (9) as

$$\rho - \frac{1}{|\mathcal{C}_L^*|} \sum_{\{\varepsilon_j^*(L)\}} \frac{1}{\mathrm{e}^{\beta(\varepsilon_j^*(L)-\mu_L(\beta,\rho))}-1} = \frac{1}{|\mathcal{C}_L^*|} \sum_{\{\epsilon_s^*(L)\}} \frac{1}{\mathrm{e}^{\beta(\epsilon_s^*(L)-\mu_L(\beta,\rho))}-1}. \qquad (11)$$

Let us now distinguish different ways in which the one-particle spectrum may behave as $L \to \infty$. We start with the generic one.

(i) Suppose that the bound-state energies $\{\epsilon_s^*(L)\}_{s\geqslant 1}$ verify the following conditions as $L \to \infty$:

$$E_1 - \delta > \epsilon_1^*(L) \qquad \text{for some } \delta > 0$$

and

$$\epsilon_s^*(L) - \epsilon_1^*(L) \geqslant a|\mathcal{C}_L^*|^{\alpha-1}, \qquad a > 0, \quad \alpha > 0.$$

This is true, in particular, for a tube with a single bend or protrusion and two cut-offs moving away from it, where in view of the (norm-resolvent) convergence of $T_*(L)$ to T_* the eigenvalue $\epsilon_1^*(L)$ tends to $\epsilon_1^* < E_1$ and the eigenvalue difference to a nonzero limit.

Since $\mu_L(\beta,\rho) < \epsilon_1^*(L)$, we can use the uniform convergence (7) to obtain the asymptotics of the solution of equation (11) as $L \to \infty$, namely

$$\mu_L(\beta,\rho) = \epsilon_1^*(L) - \frac{1}{\beta(\rho-\rho_c^*(\beta))|\mathcal{C}_L^*|} + o\left(|\mathcal{C}_L^*|^{-1}\right). \qquad (12)$$

This means that $\lim_{L\to\infty} \mu_L(\beta,\rho) = \epsilon_1^*$, and since $|\mathcal{C}_L^*| = \mathcal{O}(L)$ as $L \to \infty$, the thermodynamic limit in (11) gives rise to BEC at the lowest level ϵ_1^* only,

$$\rho - \rho_c^*(\beta) = \lim_{L\to\infty} \frac{1}{|\mathcal{C}_L^*|} \frac{1}{\mathrm{e}^{\beta(\epsilon_1^*(L)-\mu_L(\beta,\rho))}-1}. \qquad (13)$$

The same is true for other tube geometries, for instance, for a tube with several well distinguished bends or 'bubbles', as long as the limit $L \to \infty$ means that cut-offs move away in the asymptotically straight parts.

(ii) The situation may change when the thermodynamic limit $L \to \infty$ is more complicated and involves a local change of the geometry as well. As a model example, consider again a tube with a finite number $n > 1$ of well distinguished *identical* bends, but suppose now that the distances between them increase also with increasing L. As we have recalled above, the first n eigenvalues $\{\epsilon_s^*(L)\}_{s=1}^n$ then cluster, being exponentially close to each other with respect to the separation parameter L,

$$\epsilon_s^*(L) - \epsilon_1^*(L) \leqslant C\mathrm{e}^{-aL}, \qquad 1 \leqslant s \leqslant n, \tag{14}$$

for some positive C, a. By (14) the asymptotics of the solution to equation (11) is again of the form (12). This means that the limit in (11) gives rise to BEC equally fragmented into the group of n levels $\{\epsilon_s^*(L)\}_{s=1}^n$ which are almost degenerate being exponentially separated,

$$\lim_{L\to\infty} \frac{1}{|\mathcal{C}_L^*|} \frac{1}{\mathrm{e}^{\beta(\epsilon_s^*(L)-\mu_L(\beta,\rho))} - 1} = \frac{\rho - \rho_c^*(\beta)}{n} \tag{15}$$

for $1 \leqslant s \leqslant n$. This fragmentation is called a *type-I generalized* BEC , contrasting with the case of the *infinite* fragmentation known as the *type-II generalized* BEC[5], see [1] and [29, 30]. What is important is that this condensate is separated from the continuum spectrum by a finite energy gap which makes it more stable than the conventional BEC, see the discussion on this point in [14, 15].

To analyse localization properties of the geometrically induced BEC we employ the PBG *one-body reduced density matrix* with the kernel $\rho_L(\beta, \mu; x, y)$ given by

$$|\mathcal{C}_L^*| \int_{-\infty}^{\infty} \mathcal{N}_L^*(\mathrm{d}\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} \overline{\psi_{\varepsilon,L}^*(x)} \psi_{\varepsilon,L}^*(y), \tag{16}$$

where $\{\psi_{\varepsilon,L}^*\}$ are the *normalized eigenfunctions* of the operator $T_*(L)$. The diagonal part of the matrix (16) is the *local* particle density, since by (6)

$$\int_{\mathcal{C}_L^*} \mathrm{d}x\, \rho_L(\beta, \mu; x, x) = |\mathcal{C}_L^*| \rho_L(\beta, \mu) \tag{17}$$

is the total number of particles in $\mathcal{C}_L^*$. In fact, a relevant quantity for the BEC *space localization* is the *local* particle density *per unit volume* $\tilde{\rho}_L(\beta, \mu; x)$ given by

$$\int_{-\infty}^{\infty} \mathcal{N}_L^*(\mathrm{d}\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} \overline{\psi_{\varepsilon,L}^*(x)} \psi_{\varepsilon,L}^*(x). \tag{18}$$

Indeed, since in the limit only the eigenfunctions corresponding to the eigenfunction family $\{\psi_s^*\}_{s\geqslant 1} \subset L^2(\mathcal{C}^*)$ related to the infinite-tube bound states are preserved, in contrast to all the others which are extended, we get

$$\begin{aligned} \tilde{\rho}(\beta, \mu; x) &:= \lim_{L\to\infty} \int_{-\infty}^{\infty} \mathcal{N}_L^*(\mathrm{d}\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} \overline{\psi_{\varepsilon,L}^*(x)} \psi_{\varepsilon,L}^*(x) \\ &= (\rho - \rho_c^*(\beta)) \overline{\psi_{\epsilon_1}^*(x)} \psi_{\epsilon_1}^*(x) \end{aligned} \tag{19}$$

for $\rho > \rho_c^*(\beta)$ in the case (13). For the fragmented BEC (15) one gets for $\rho > \rho_c^*(\beta)$ similarly

$$\tilde{\rho}(\beta, \mu; x) = \frac{\rho - \rho_c^*(\beta)}{n} \sum_{s=1}^{n} |\psi_{\epsilon_s}^*(x)|^2; \tag{20}$$

it is obvious that $\tilde{\rho}(\beta, \mu; x) = 0$ holds when $\rho \leqslant \rho_c^*(\beta)$.

[5] We do not discuss here the intermediate case, $\epsilon_s^*(L) - \epsilon_1^*(L) = a_s |\mathcal{C}_L^*|^{-1-\alpha}$ with $a > 0$ and $\alpha \geqslant 0$, when the so-called *type-II* ($\alpha = 0$) *and type-III* ($\alpha > 0$) *generalized* BEC can take place [29, 30]. In these situations none of the bound states is macroscopically occupied. This can occur with geometrically deformed tubes only for complicated shapes which are rather mathematical constructs without a relation to a possible experiment.

Thus in contrast to the BEC in the translation-invariant case, the L^2-localized condensation in the bounded states corresponds to an *infinite* accumulation of the local particle density defined by (16). On the other hand, by (13) and (15) in combination with (19), (20), the *total* density of particles condensed in the bounded states,

$$\rho - \rho_c^*(\beta) = \int_{\mathcal{C}^*} \mathrm{d}x\, \tilde{\rho}(\beta, \mu; x), \tag{21}$$

is *finite*.

Summarizing the above analysis, the predicted localized one-dimensional BE condensation in the perfect Bose gas is due to bound states separated from the continuum. One asks naturally whether this effect is stable with respect to the particle interactions. The answer can be given in the case of a mean-field repulsive interaction, i.e. in the situation when the two-body particle potential v has the form

$$v(|x-y|) = \frac{\lambda}{|\mathcal{C}_L^*|}, \qquad \lambda > 0. \tag{22}$$

Consider for simplicity a single bend (or protrusion) corresponding to an isolated bound state with the energy ϵ_1^*. Then by [15] and by the definition (4) we obtain that the condensate density in the state $\psi_{\epsilon_1}^*$ is given by

$$\rho_0^{\mathrm{MF}}(\beta, \mu) = \begin{cases} 0 & \text{for} \quad \mu \leqslant \epsilon_1^* + \lambda\rho_c^*(\beta) \\ \dfrac{\mu - \epsilon_1^*}{\lambda} - \rho_c^*(\beta) & \text{for} \quad \mu > \epsilon_1^* + \lambda\rho_c^*(\beta), \end{cases} \tag{23}$$

so the effect persists. On the other hand, in view of the remark preceding (21) the stability of the L^2-localized BE condensate is much less evident in the case of the non-mean-field two-body interaction, see the discussion in [15] and references there; the question deserves a separate investigation.

After this theoretical analysis let us ask about the chances of observing the described type of the BE condensation in an experiment. We are not going to discuss technically the ways in which the Bose gas can be confined in a geometrically deformed tube; generally we have in mind either a modification of the existing elongated traps—see recent experiments of the MIT group [31, 32]—or using hollow optical fibres—see, e.g., [33] and references therein—as suggested in [34].

The important parameter is the tube radius which for both the elongated traps and hollow fibres can be made as small as $r \approx 5\ \mu$m which determines the threshold energy by the mentioned expression $E_1 = \frac{\hbar^2}{2M} j_{0,1}^2 r^{-2}$, where M is the atom of the mass in question. Let us further introduce the *relative gap* size by $\gamma_{\mathrm{rel}} := (E_1 - \epsilon_1^*)/E_1$. This quantity can theoretically reach the value of about 0.39 in a smoothly bent tube [35], but a typical value[6] for a bending angle of 90° and more is $\gamma_{\mathrm{rel}} \approx 10^{-1}$, see [28].

Two conditions must be satisfied. First of all, the bending-induced gap $\gamma_{\mathrm{rel}} E_1$ must be much larger than the effect of the finite length L of the recipient. The latter is characterized by the first longitudinal eigenvalue $\frac{\hbar^2}{2M}(\pi/L)^2$. It is clear that the trap must be sufficiently elongated to fulfil the condition, roughly speaking $L/r \gtrsim 20$ is sufficient. This is true in both situations mentioned above; for hollow fibres it can be even better.

More complicated is the question of thermal stability; the said gap must be larger than the energy $k_{\mathrm{B}}T$. This determines a critical temperature above which the bending-induced BEC is likely to be destroyed by thermal fluctuations. Using the value $r \approx 5\mu$ m we find that

$$T_{\mathrm{crit}} \approx 5.4 \times 10^{-8} \frac{\gamma_{\mathrm{rel}}}{Z}, \tag{24}$$

[6] For other geometries this parameter can be made larger: the upper bound for crossed wires is 0.75, see [26], while in protruded tubes γ_{rel} can be arbitrarily close to 1.

where Z is the atomic number in question. Hence lighter nuclei are preferable; there is one order of magnitude difference between Li^7 and Ru^{87}. With the above estimate in mind, however, even for the light ones it is difficult to perform the measurement in the available nanokelvin conditions. On the other hand, the effect is not too far from experimental reach; recall for instance that squeezing the transverse size to a 1 μm radius would enhance the critical temperature (24) by a factor of 25.

Let us finally mention one more feature of such a geometrically induced BE condensate. The ground-state wavefunction into which the atoms of the Bose gas will condense (cf (19)) is exponentially localized away from the bend; for examples of such wavefunctions see [28] and the literature mentioned there. This means that, e.g., bending a tube would mean not only the condensation but also that the more the condensate is squeezed in the bend the larger will be the bending angle. This property distinguishes the condensate discussed here from the other quasi-one-dimensional systems considered before.

In conclusion, we have demonstrated here a purely geometric way to achieve a stable BEC based on local deformations of a tube-shaped recipient, and we have discussed ways in which the effect could be experimentally observed.

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